



TITLE:

# Application of the lace expansion to the $\varphi^4$ model (Probability Symposium)

AUTHOR(S):

Sakai, Akira

---

CITATION:

Sakai, Akira. Application of the lace expansion to the  $\varphi^4$  model (Probability Symposium). 数理解析研究所講究録 2013, 1855: 80-81

ISSUE DATE:

2013-10

URL:

<http://hdl.handle.net/2433/195227>

RIGHT:

## Application of the lace expansion to the $\varphi^4$ model

Akira Sakai<sup>1</sup>

Department of Mathematics  
Hokkaido University

The  $\varphi^4$  model is a standard model in scalar field theory. It is defined as the Gaussian free field combined with quartic self-interaction. Let  $\Lambda \subset \mathbb{Z}^d$  and define the Hamiltonian for the spin configuration  $\varphi = \{\varphi_x\}_{x \in \Lambda}$  as

$$\mathcal{H}_\Lambda(\varphi) = - \sum_{\{u,v\} \subset \Lambda} \mathcal{J}_{u,v} \varphi_u \varphi_v + \sum_{v \in \Lambda} \left( \frac{\mu}{2} \varphi_v^2 + \frac{\lambda}{4!} \varphi_v^4 \right),$$

where  $\mu \in \mathbb{R}$  plays the role of the temperature, while the intensity  $\lambda$  of self-interaction is fixed nonnegative. We assume that the spin-spin coupling  $\mathcal{J}_{u,v}$  is ferromagnetic (i.e.,  $\mathcal{J}_{u,v} \geq 0$ ), translation-invariant (i.e.,  $\mathcal{J}_{u,v} = \mathcal{J}_{o,v-u}$ ),  $\mathbb{Z}^d$ -symmetric and finite-range. For example, the nearest-neighbor coupling  $\mathcal{J}_{o,x} = \delta_{|x|,1}$  satisfies all those properties. Let

$$\langle \varphi_o \varphi_x \rangle_\mu = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\int_{\mathbb{R}^\Lambda} \varphi_o \varphi_x e^{-\mathcal{H}_\Lambda(\varphi)} d^\Lambda \varphi}{\int_{\mathbb{R}^\Lambda} e^{-\mathcal{H}_\Lambda(\varphi)} d^\Lambda \varphi}.$$

It is known to exhibit a phase transition and critical behavior: there is a  $\mu_c = \mu_c(d, \mathcal{J}, \lambda)$ , which is not larger than the critical point  $\hat{\mathcal{J}} \equiv \sum_{x \in \mathbb{Z}^d} \mathcal{J}_{o,x}$  for the Gaussian free field, such that  $\chi_\mu \equiv \sum_{x \in \mathbb{Z}^d} \langle \varphi_o \varphi_x \rangle_\mu$  is finite if and only if  $\mu > \mu_c$  and diverges as  $\mu \downarrow \mu_c$  [7]. There were intensive researches in the 1980's when Aizenman [1] and Fröhlich [2] succeeded in showing mean-field behavior (e.g.,  $\chi_\mu$  is bounded above and below by a positive multiple of  $(\mu - \mu_c)^{-1}$  as  $\mu \downarrow \mu_c$ ) above 4 dimensions under the assumption of reflection-positivity. The nearest-neighbor model satisfies this assumption. In 4 dimensions, Gawędzki and Kupiainen [4] and Hara and Tasaki [5, 6] succeeded in showing the mean-field behavior (with log corrections) for the weakly coupled nearest-neighbor model using a rigorous renormalization-group method.

The sufficient condition for the mean-field behavior that Aizenman suggested in [1] is the bubble condition

$$\sum_{x \in \mathbb{Z}^d} \langle \varphi_o \varphi_x \rangle_{\mu_c}^2 < \infty.$$

For reflection-positive models, the Fourier transform of  $\langle \varphi_o \varphi_x \rangle_\mu$  is known to obey the Gaussian infrared bound [3]

$$0 \leq \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} \langle \varphi_o \varphi_x \rangle_\mu \leq O(|k|^{-2}) \quad \text{uniformly in } \mu > \mu_c,$$

which implies that the bubble condition holds for  $d > 4$ , hence the mean-field behavior. Although the result is satisfactory, it is often hard to verify the assumption of reflection-positivity.

The goal of my research is to investigate asymptotic behavior of the critical two-point function  $\langle \varphi_o \varphi_x \rangle_{\mu_c}$  above the upper-critical dimension, without the assumption of reflection-positivity. In [9], we prove the following:

<sup>1</sup><http://www.math.sci.hokudai.ac.jp/~sakai/>

**Theorem 1.** Let  $\rho = 2(d - 4) > 0$  and  $0 < \lambda \ll 1$  (depending on  $d$  and  $\mathcal{J}$ ). Then there is a  $\Phi_\mu(x) = \langle \varphi_o^2 \rangle_\mu \delta_{o,x} + O(\lambda) (|x| \vee 1)^{-(d+2+\rho)}$ , where  $O(\lambda)$  is uniform in  $\mu \geq \mu_c$ , such that

$$\mu_c = \hat{\mathcal{J}} - \frac{\lambda}{2} \hat{\Phi}_{\mu_c}, \quad \langle \varphi_o \varphi_x \rangle_{\mu_c} \underset{|x| \uparrow \infty}{\sim} \frac{\frac{d}{2} \Gamma(\frac{d-2}{2}) \pi^{-d/2}}{\sum_{y \in \mathbb{Z}^d} |y|^2 (\mathcal{J}_{o,y} - \frac{\lambda}{2} \Phi_{\mu_c}(y))} |x|^{2-d}.$$

The key elements for the proof of the above theorem are the following:

1. The Griffiths-Simon construction [10] to approximate the  $\varphi^4$  model on  $\Lambda$  to some Ising model on  $\Lambda \times \{1, 2, \dots, N\}$ .
2. The lace expansion for the Ising two-point function [8].
3. Detail estimates on the expansion coefficients in terms of  $N$  [9].

These steps yield a linearized version of the Schwinger-Dyson equation, which further yields the aforementioned asymptotic expression for  $\langle \varphi_o \varphi_x \rangle_{\mu_c}$ .

## References

- [1] M. Aizenman, *Geometric analysis of  $\phi^4$  fields and Ising models*, Comm. Math. Phys. **86** (1982), 1–48.
- [2] J. Fröhlich, *On the triviality of  $\lambda\varphi_d^4$  theories and the approach to the critical point in  $d \geq 4$  dimensions*, Nucl. Phys. B **200** [FS4] (1982), 281–296.
- [3] J. Fröhlich, B. Simon, T. Spencer, *Infrared bounds, phase transitions and continuous symmetry breaking*, Comm. Math. Phys. **50** (1976), 79–95.
- [4] K. Gawędzki and A. Kupiainen, *Massless lattice  $\phi_4^4$  theory: rigorous control of a renormalizable asymptotically free model*, Comm. Math. Phys. **99** (1985), 197–252.
- [5] T. Hara, *A rigorous control of logarithmic corrections in four-dimensional  $\varphi^4$  spin systems. I. Trajectory of effective Hamiltonians*, J. Stat. Phys. **47** (1987), 57–98.
- [6] T. Hara and H. Tasaki, *A rigorous control of logarithmic corrections in four-dimensional  $\varphi^4$  spin systems. II. Critical behavior of susceptibility and correlation length*, J. Stat. Phys. **47** (1987), 99–121.
- [7] J. Lebowitz, *GHS and other inequalities*, Comm. Math. Phys. **35** (1974), 87–92.
- [8] A. Sakai, *Lace expansion for the Ising model*, Comm. Math. Phys. **272** (2007), 283–344.
- [9] A. Sakai, *Application of the lace expansion to the  $\varphi^4$  model*, In preparation.
- [10] B. Simon and R.B. Griffiths, *The  $(\phi^4)_2$  field theory as a classical Ising model*, Comm. Math. Phys. **33** (1973), 145–164.